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Interpolation of Fremlin tensor products and Schur factorization of matrices

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Abstract

Within finite dimensional Banach lattices we prove interpolation formulas for the Fremlin tensor product and spaces of regular multilinear forms and operators. We show applications to factorization of matrices with respect to the Schur product. Our results imply various abstract variants of Schur's classical result, and in particular we extend Pisier's converse for matrices in finite dimensional ℓ_p-spaces to the setting of complex Calderón interpolation of finite dimensional Banach lattices.

Keywords: Banach lattice; Complex interpolation; Fremlin tensor product; Regular operator; Calderón–Lozanovskii space

1. Introduction

In the theory of matrices as well as operators the Schur product plays a significant role. Throughout this article M_n denotes all complex $n \times n$-matrices. For two matrices $A = [a_{ij}]$ and $B = [b_{ij}] \in M_n$ the Schur product is defined by

$$A * B = [a_{ij} b_{ij}].$$

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Note that M_n with this product is an algebra. Here we also consider the lattice structure on M_n given by the pointwise order, i.e., the modulus of A is given by $|A| = [|a_{ij}|]$, and the order relation $|A| \leqslant |B|$ means that $|a_{ij}| \leqslant |b_{ij}|$ for each $i, j \in \{1, \dots, n\}$. In what follows $|A|^s := [|a_{ij}|^s]$ for every $A = [a_{ij}] \in M_n$ and every $s > 0$.

The following factorization theorem of matrices in terms of the Schur product is a prototypical result.

Theorem 1.1. *Let $0 < \theta < 1$ and $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ where $1 \leqslant p_0, p_1 \leqslant \infty$. Then for any $A \in M_n$ the following two statements are equivalent*:

(i) *There exist $A_0, A_1 \in M_n$ such that $|A| = |A_0|^{1-\theta} * |A_1|^\theta$ and*

$$\bigl\| |A_0| : \ell_{p_0}^n \to \ell_{p_0}^n \bigr\| \leqslant 1, \qquad \bigl\| |A_1| : \ell_{p_1}^n \to \ell_{p_1}^n \bigr\| \leqslant 1.$$

(ii) $\bigl\| |A| : \ell_{p_\theta}^n \to \ell_{p_\theta}^n \bigr\| \leqslant 1.$

Here, as usual, the Banach space ℓ_p^n denotes the vector space $\mathbb{C}^n$ equipped with the p-norm $\|\cdot\|_p$, and $\|A\|$ stands for the operator norm of a linear operator (matrix) A acting on ℓ_p^n. In the above theorem the proof of the implication (i) $\Rightarrow$ (ii) is the easy one. For the special case $p_0 = 1$, $p_1 = \infty$ and $\theta = 1/2$ (hence $p_\theta = 2$) the statement of this implication is called Schur criterion, and was published (precisely one century ago) in [15, Satz I]. Schur's proof extends to the more general situation presented here. The implication (ii) $\Rightarrow$ (i) is less trivial. Again the specific case $p_0 = 1$, $p_1 = \infty$ is known and part of Pisier's remarkable paper [12, Theorem 1].

The preceding factorization theorem has a very condensed formulation in terms of complex interpolation. In fact, if $\mathscr{L}^r(X)$ stands for the Banach space of all linear operators on a finite dimensional Banach lattice X equipped with the so called regular norm $\|A\|_r = \||A| : X \to X\|$, and if for every $0 < \theta < 1$ we denote by $[X_0, X_1]_\theta$ the complex interpolation space with respect to a couple (X_0, X_1) of finite dimensional Banach lattices, then the above theorem exactly means that with equality of norms we have

$$\bigl[\mathscr{L}^r\bigl(\ell_{p_0}^n\bigr), \mathscr{L}^r\bigl(\ell_{p_1}^n\bigr)\bigr]_\theta = \mathscr{L}^r\bigl(\ell_{p_\theta}^n\bigr), \quad 0 < \theta < 1. \tag{1}$$

One of our main results is the following far reaching extension of this interpolation formula.

Theorem 1.2. *Let (X_0, X_1) and (Y_0, Y_1) be two couples of finite dimensional Banach lattices, and $0 < \theta < 1$. Then for every matrix $A \in M_n$ with*

$$\bigl\| |A| : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta \bigr\| \leqslant 1$$

and every $1 \leqslant p, q \leqslant \infty$ there exist matrices A_0, $A_1 \in M_n$ such that

$$|A| = |A_0|^{1-\theta} * |A_1|^\theta \tag{2}$$

and

$$\max_{k=0,1} \bigl\| |A_k| : X_k \to Y_k \bigr\| \leqslant \bigl(M^p(X_0)^{1-\theta} M^q(X_1)^\theta\bigr)\bigl(M_p(Y_0)^{1-\theta} M_q(Y_1)^\theta\bigr).$$

Alternatively, the following different norm estimates hold: For $1 < p, q < \infty$ *with* $1/p + 1/q \leqslant 1$ *there is a factorization of* A *given by the formula* (2)*, and*

$$\max_{k=0,1} \bigl\| |A_k| : X_k \to Y_k \bigr\| \leqslant \bigl(M^p(X_0)^{1-\theta} M^p(X_1)^\theta\bigr)\bigl(M_{q'}(Y_0)^{1-\theta} M_{q'}(Y_1)^\theta\bigr).$$

Here $M^p(X)$ and $M_p(X)$ denote the p-convexity and p-concavity constant, respectively, of a Banach lattice X.

The proof of the above theorem is based on complex interpolation formulas for couples $(\bigotimes_{|\pi|} X_k, \bigotimes_{|\pi|} Y_k)$ of Fremlin tensor products of finite dimensional Banach lattices:

$$\bigl[\bigotimes\nolimits_{|\pi|} X_k, \bigotimes\nolimits_{|\pi|} Y_k\bigr]_\theta = \bigotimes\nolimits_{|\pi|} [X_k, Y_k]_\theta, \quad 0 < \theta < 1. \tag{3}$$

Applying such formulas and using duality, we also obtain interpolation formulas for couples $(\mathscr{B}^r(X_1, \ldots, X_m), \mathscr{B}^r(Y_1, \ldots, Y_m))$ of spaces of regular multilinear forms:

$$\bigl[\mathscr{B}^r(X_1, \ldots, X_m), \mathscr{B}^r(Y_1, \ldots, Y_m)\bigr]_\theta = \mathscr{B}^r\bigl([X_1, Y_1]_\theta, \ldots, [X_m, Y_m]_\theta\bigr), \quad 0 < \theta < 1.$$

Our results of this type extend the formula from (1) on spaces of regular operators in finite dimensional ℓ_p-spaces.

It should be pointed out here that in the case when $m = 2$ interpolation formulas of this type for Banach spaces instead of Banach lattices and projective tensor products instead of Fremlin tensor products were studied by Kouba [9] (see also [6,7]).

We remark that the two different norm estimates in Theorem 1.2 are consequences of our two different approaches to formula (3). The first case is based on Schep's atomic description of Fremlin's tensor products. The second one uses variants of the Maurey–Rosenthal factorization theorem for multilinear regular forms on Banach lattices. We note that both approaches involve different convexity assumptions on the spaces $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$. Some of our interpolation formulas are extended to the Calderón–Lozanovskii construction.

2. Preliminaries

We shall use standard notation and notions from Banach space theory, as presented, e.g., in [10] or [16]; for tensor products of Banach spaces we refer to [4]. All Banach spaces we deal with are complex as well as finite dimensional. An n-dimensional complex Banach space $X = (\mathbb{C}^n, \|\cdot\|_X)$ is said to be a lattice if $\|\cdot\|_X$ is a lattice norm: $|x| \leqslant |y|$ implies $\|x\|_X \leqslant \|y\|_X$. Following Calderón [2], for two n-dimensional Banach lattices X_0, X_1 and $0 < \theta < 1$ the Banach lattice $X_0^{1-\theta} X_1^\theta$ denotes $\mathbb{C}^n$ equipped with the norm

$$\|x\|_{X_0^{1-\theta} X_1^\theta} = \inf\bigl\{\|x_0\|_{X_0}^{1-\theta} \|x_1\|_{X_1}^\theta \colon |x| = |x_0|^{1-\theta} |x_1|^\theta,\ x_0, x_1 \in \mathbb{C}^n\bigr\}.$$

Observe that for an n-dimensional Banach lattice X and $0 < \theta < 1$, we have with equality of norms the following formula

$$X^{1-\theta} \bigl(\ell_\infty^n\bigr)^\theta = X^p, \quad p = 1/(1-\theta), \tag{4}$$

where X^p denotes the p-convexification of X. Given $1 \leqslant r \leqslant \infty$ we denote the r-convexity (respectively, r-concavity) constant of a finite dimensional Banach lattice X by $M^r(X)$ (respectively, $M_r(X)$), i.e., the smallest constant $C > 0$ such that for every finite sequence $\{x_1, \dots, x_n\}$ in X,

$$\left\| \left(\sum_{k=1}^{n} |x_k|^r \right)^{1/r} \right\| \leqslant C \left(\sum_{k=1}^{n} \|x_k\|^r \right)^{1/r},$$

respectively,

$$\left(\sum_{k=1}^{n} \|x_k\|^r \right)^{1/r} \leqslant C \left\| \left(\sum_{k=1}^{n} |x_k|^r \right)^{1/r} \right\|$$

(with the obvious modification whenever $r = \infty$). Recall that $M^r(X)$ and $M_r(X)$ are nondecreasing, respectively, nonincreasing functions of r. We will often use without any further reference the following duality relation:

$$M_r(X) = M^{r'}\big(X^*\big), \qquad 1/r' := 1 - 1/r.$$

It is well known that for all $1 \leqslant r, p \leqslant \infty$

$$M^r\big(\ell_p^n\big) = n^{\max\{0, 1/p - 1/r\}} \quad \text{and} \quad M_r\big(\ell_p^n\big) = n^{\max\{0, 1/r - 1/p\}}. \tag{5}$$

A very much simplifying fact will be that for each $1 \leqslant r < \infty$ and every finite dimensional Banach lattice X there is a norm p_X on X such that $M^r((X, p_X)) = 1$ and

$$M^r(X)^{-1} \|x\|_X \leqslant p_X(x) \leqslant \|x\|_X, \quad x \in X. \tag{6}$$

We will frequently use the following simple estimate (see, e.g., [16, pp. 218–219]):

$$M^r\big(X_0^{1-\theta} X_1^{\theta}\big) \leqslant M^r(X_0)^{1-\theta} M^r(X_1)^{\theta}. \tag{7}$$

For details on complex interpolation we refer to [1,2]. Given an interpolation couple (E_0, E_1) of complex Banach spaces and $0 < \theta < 1$, the complex interpolation space with respect to (E_0, E_1) is denoted by $[E_0, E_1]_\theta$. In what follows by a finite dimensional interpolation couple (E_0, E_1), we always mean a finite dimensional linear space equipped with two norms. We will heavily use the following complex interpolation formulas due to Calderón [2]: For every interpolation couple (X_0, X_1) of finite dimensional Banach lattices and $0 < \theta < 1$ the following formulas hold with equality of norms:

$$[X_0, X_1]_\theta = X_0^{1-\theta} X_1^{\theta} \tag{8}$$

and

$$[X_0, X_1]_\theta^* = \big[X_0^*, X_1^*\big]_\theta. \tag{9}$$

In this article we study Fremlin tensor products of Banach lattices. Recall that for finite dimensional Banach lattices $X_1, \dots, X_m$ the Fremlin projective tensor norm of $u \in \bigotimes X_k$ is given by

$$\|u\|_{|\pi|} = \inf\left\{ \sum_{j=1}^{n} \prod_{k=1}^{m} \|x_j^k\|_{X_k} \colon x_j^k \in X_k \text{ such that } |u| \leqslant \sum_{j=1}^{n} x_j^1 \otimes \cdots \otimes x_j^m \right\}.$$

The Banach lattice $\bigotimes_{|\pi|} X_k := (\bigotimes X_k, \|\cdot\|_{|\pi|})$ is called the Fremlin tensor product of the $X_1, \dots, X_m$. In what follows we write $X \otimes_{|\pi|} Y$ in the case of two Banach lattices. For all needed information on Fremlin tensor products we refer to [8] and [14].

The regular norm $\|T\|_r$ of a linear operator $T = [a_{ij}] : X \to Y$ between finite dimensional Banach lattices X and Y is defined to be the operator norm $\||T|\|$ of the modulus $|T| = [|a_{ij}|]$ of T. The space $\mathscr{L}(X, Y)$ of all linear operators equipped with the norm $\|\cdot\|_r$ is a Banach lattice, and as usual denoted by $\mathscr{L}^r(X, Y)$. It is well known that the canonical mapping

$$\mathscr{L}^r\big(X, Y^*\big) \to (X \otimes_{|\pi|} Y)^*, \quad T \mapsto \big[x \otimes y \mapsto T(x)(y)\big] \tag{10}$$

constitutes a lattice isometric homomorphism (see [8]). Similarly we define the space $\mathscr{B}^r(X_1, \dots, X_m)$, the Banach lattice of all m-linear forms $\varphi = [a_{j_1,\dots,j_m}]$ on $\prod_{k=1}^m X_k$ endowed with the norm $\|\varphi\|_r = \||\varphi|\|$. As an analog of (10) we in this case have the following lattice isometric homomorphism (see [14]):

$$\mathscr{B}^r(X_1, \dots, X_m) = \big(\bigotimes\nolimits_{|\pi|} X_k\big)^*. \tag{11}$$

3. Interpolation of regular forms and operators

We start with the following theorem which will be one of the crucial tools in our study.

Theorem 3.1. *Let (X_k, Y_k), $1 \leqslant k \leqslant m$, be couples of finite dimensional Banach lattices. Then for every $0 < \theta < 1$ we have*

$$\big\|\mathrm{id} : \bigotimes\nolimits_{|\pi|}[X_k, Y_k]_\theta \to \big[\bigotimes\nolimits_{|\pi|} X_k, \bigotimes\nolimits_{|\pi|} Y_k\big]_\theta\big\| \leqslant 1,$$

and

$$\big\|\mathrm{id} : \big[\mathscr{B}^r(X_1, \dots, X_m), \mathscr{B}^r(Y_1, \dots, Y_m)\big]_\theta \to \mathscr{B}^r\big([X_1, Y_1,]_\theta, \dots, [X_m, Y_m]_\theta\big)\big\| \leqslant 1.$$

Proof. We start with the proof of the first norm estimate. First observe that the m-linear mappings

$$\big(\bigotimes\nolimits_{|\pi|} X_k\big)^* \times X_1 \times \dots X_m \to \mathbb{C}, \quad \big(\bigotimes x_k^*, x_1, \dots, x_m\big) \mapsto \prod_{k=1}^m x_k^*(x_k),$$

$$\big(\bigotimes\nolimits_{|\pi|} Y_k\big)^* \times Y_1 \times \dots Y_m \to \mathbb{C}, \quad \big(\bigotimes y_k^*, y_1, \dots, y_m\big) \mapsto \prod_{k=1}^m y_k^*(y_k)$$

are both positive and have norm $\leqslant 1$. Hence by multilinear complex interpolation (see [1,2]) we obtain that the bilinear mapping

$$\bigl[\bigl(\bigotimes_{|\pi|} X_k\bigr)^*, \bigl(\bigotimes_{|\pi|} Y_k\bigr)^*\bigr]_\theta \times \prod_{k=1}^m [X_k, Y_k]_\theta \to \mathbb{C}, \quad \Bigl(\bigotimes z_k^*, (z_1, \ldots, z_m)\Bigr) \mapsto \prod_{k=1}^m z_k^*(z_k)$$

is positive with norm $\leqslant 1$. Hence by dualization and linearization, we conclude that the same holds for

$$\bigl[\bigotimes_{|\pi|} X_k, \bigotimes_{|\pi|} Y_k\bigr]_\theta^* \times \bigotimes_{|\pi|} [X_k, Y_k]_\theta \to \mathbb{C}, \quad \Bigl(\bigotimes z_k^*, \bigotimes_k z_k\Bigr) \mapsto \prod_{k=1}^m z_k^*(z_k),$$

which yields the required estimate

$$\Bigl\| \mathrm{id} : \bigotimes_{|\pi|} [X_k, Y_k]_\theta \to \bigl[\bigotimes_{|\pi|} X_k, \bigotimes_{|\pi|} Y_k\bigr]_\theta^{**} \Bigr\| \leqslant 1.$$

The proof of the second statement is a consequence of the first statement and the duality formula (10). □

4. Interpolation of couples of Fremlin tensor products

Below we state the main theorem of this section. Later these interpolation formulas will be used to prove variants of Schur's classical criterion mentioned in the introduction.

Theorem 4.1. *Let (X_0, X_1) and (Y_0, Y_1) be couples of finite dimensional Banach lattices, and $1 \leqslant p, q \leqslant \infty$. Then*

(i) *If $M^p(X_0) = M^{p'}(Y_0) = M^q(X_1) = M^{q'}(Y_1) = 1$, then for every $0 < \theta < 1$ with equality of norms*

$$[X_0 \otimes_{|\pi|} Y_0, X_1 \otimes_{|\pi|} Y_1]_\theta = [X_0, X_1]_\theta \otimes_{|\pi|} [Y_0, Y_1]_\theta.$$

(ii) *If $M^p(X_0) = M_p(Y_0) = M^q(X_1) = M_q(Y_1) = 1$, then for every $0 < \theta < 1$ with equality of norms*

$$\bigl[\mathscr{L}^r(X_0, Y_0), \mathscr{L}^r(X_1, Y_1)\bigr]_\theta = \mathscr{L}^r\bigl([X_0, X_1]_\theta, [Y_0, Y_1]_\theta\bigr).$$

The proof needs the two Lemmas 4.2 and 4.3. Note that for couples of Fremlin tensor products the second lemma complements Theorem 3.1, and gives an immediate proof of the preceding theorem.

The following notion is motivated by Schep's article [14], and it is crucial for the first lemma: For a given Fremlin tensor product $X \otimes_{|\pi|} Y$ of two finite dimensional Banach lattices, we denote by $C(X \otimes_{|\pi|} Y)$ the least constant $C > 0$ such that for all $u \in X \otimes_{|\pi|} Y$ we have

$$\inf\bigl\{\|x\|\|y\|: |u| \leqslant x \otimes y, \ x \in X, \ y \in Y\bigr\} \leqslant C\|u\|_{X \otimes_{|\pi|} Y};$$

this constant will be called the atomic constant of $X \otimes_{|\pi|} Y$.

Based on a straight forward analysis of [14, Theorems 2.1, 2.2] we obtain the following.

Lemma 4.2. *Let X and Y be two finite dimensional Banach lattices. Then for every $1 \leqslant p \leqslant \infty$*

$$C(X \otimes_{|\pi|} Y) \leqslant M^p(X) M^{p'}(Y),$$

where $1/p + 1/p' = 1$.

This variant of Schep's results is essential for the proof of our second lemma.

Lemma 4.3. *Let (X_0, X_1) and (Y_0, Y_1) be two couples of finite dimensional Banach lattices, and $1 \leqslant p, q \leqslant \infty$. Then for each $0 < \theta < 1$ we have*

$$\begin{aligned}&\left\| \mathrm{id} : [X_0 \otimes_{|\pi|} Y_0, X_1 \otimes_{|\pi|} Y_1]_\theta \to [X_0, X_1]_\theta \otimes_{|\pi|} [Y_0, Y_1]_\theta \right\| \\ &\quad \leqslant \left(M^p(X_0) M^{p'}(Y_0)\right)^{1-\theta} \left(M^q(X_1) M^{q'}(Y_1)\right)^\theta,\end{aligned}$$

and

$$\begin{aligned}&\left\| \mathrm{id} : \mathscr{L}^r\left([X_0, X_1]_\theta, [Y_0, Y_1]_\theta\right) \to \left[\mathscr{L}^r(X_0, Y_0), \mathscr{L}^r(X_1, Y_1)\right]_\theta \right\| \\ &\quad \leqslant \left(M^p(X_0) M_p(Y_0)\right)^{1-\theta} \left(M^q(X_1) M_q(Y_1)\right)^\theta.\end{aligned}$$

Proof. Without loss of generality (see (6)) we may assume that all involved convexity constants are 1. Fix

$$u \in [X_0 \otimes_{|\pi|} Y_0, X_1 \otimes_{|\pi|} Y_1]_\theta = (X_0 \otimes_{|\pi|} Y_0)^{1-\theta} (X_1 \otimes_{|\pi|} Y_1)^\theta$$

with

$$\|u\|_{(X_0 \otimes_{|\pi|} Y_0)^{1-\theta} (X_1 \otimes_{|\pi|} Y_1)^\theta} < 1.$$

We claim that $\|u\|_{X_0^{1-\theta} X_1^\theta \otimes_{|\pi|} Y_0^{1-\theta} Y_1^\theta} < 1$. Indeed, by definition we see that there are $u_0 \in X_0 \otimes_{|\pi|} Y_0$ and $u_1 \in X_1 \otimes_{|\pi|} Y_1$ for which

$$|u| \leqslant |u_0|^{1-\theta} |u_1|^\theta \quad \text{and} \quad \|u_0\|_{X_0 \otimes_{|\pi|} Y_0}, \|u_1\|_{X_1 \otimes_{|\pi|} Y_1} < 1.$$

On the other hand the preceding lemma implies that there are $x_k \in X_k$, $y_k \in Y_k$ such that $|u_k| \leqslant x_k \otimes y_k$ and $\|x_k\|_{X_k} \|y_k\|_{Y_k} < 1$ for $k = 0, 1$. Hence

$$|u| \leqslant (x_0 \otimes y_0)^{1-\theta} (x_1 \otimes y_1)^\theta = x_0^{1-\theta} x_1^\theta \otimes y_0^{1-\theta} y_1^\theta,$$

and then

$$\begin{aligned}\|u\|_{X_0^{1-\theta}X_1^{\theta}\otimes_{|\pi|}Y_0^{1-\theta}Y_1^{\theta}} &\leqslant \|x_0^{1-\theta}x_1^{\theta}\|_{X_0^{1-\theta}X_1^{\theta}}\|y_0^{1-\theta}y_1^{\theta}\|_{Y_0^{1-\theta}Y_1^{\theta}}\\ &\leqslant \big(\|x_0\|_{X_0}^{1-\theta}\|x_1\|_{X_1}^{\theta}\big)\big(\|y_0\|_{Y_0}^{1-\theta}\|y_1\|_{Y_1}^{\theta}\big)\\ &\leqslant \big(\|x_0\|_{X_0}\|y_0\|_{Y_0}\big)^{1-\theta}\big(\|x_1\|_{X_1}\|y_1\|_{Y_1}\big)^{\theta} < 1,\end{aligned}$$

which completes the proof of the first inequality. The second statement follows by duality from (10). □

Using (5) we deduce from the preceding theorem a corollary for finite dimensional ℓ_p-spaces, which was proved by Pisier in [12, Theorem 1] for the case $p_0 = \infty$, $q_0 = 1$ and $p_1 = 1, q_1 = \infty$.

Corollary 4.4. *For $k = 0, 1$ let $1 \leqslant p_k, q_k \leqslant \infty$, and let $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$ with $0 < \theta < 1$.*

(i) *If $1/p_0 + 1/q_0 \leqslant 1$ and $1/p_1 + 1/q_1 \leqslant 1$, then for each n the following formula holds with equality of norms*

$$\big[\ell_{p_0}^n \otimes_{|\pi|} \ell_{q_0}^n, \ell_{p_1}^n \otimes_{|\pi|} \ell_{q_1}^n\big]_\theta = \ell_{p_\theta}^n \otimes_{|\pi|} \ell_{q_\theta}^n.$$

(ii) *If $p_k \leqslant q_k$ for $k = 0, 1$, then for each n the following formula holds with equality of norms*

$$\big[\mathscr{L}^r\big(\ell_{p_0}^n, \ell_{q_0}^n\big), \mathscr{L}^r\big(\ell_{p_1}^n, \ell_{q_1}^n\big)\big]_\theta = \mathscr{L}^r\big(\big[\ell_{p_0}^n, \ell_{p_1}^n\big]_\theta, \big[\ell_{q_0}^n, \ell_{q_1}^n\big]_\theta\big).$$

The following counterexample shows that the interpolation formula given in (i) is not true in general provided one of the two conditions fails.

Example 4.5.

$$\sup_n \big\|\mathrm{id} : \big[\ell_\infty^n \otimes_{|\pi|} \ell_\infty^n, \ell_1^n \otimes_{|\pi|} \ell_1^n\big]_{\frac{1}{2}} \to \ell_2^n \otimes_{|\pi|} \ell_2^n\big\| = \infty.$$

Proof. Denote the above supremum by C, and assume that $C < \infty$. Then we deduce from (8) that with constants independent of n we have

$$\big(\ell_\infty^n \otimes_{|\pi|} \ell_\infty^n\big)^{\frac{1}{2}}\big(\ell_1^n \otimes_{|\pi|} \ell_1^n\big)^{\frac{1}{2}} = \ell_2^n \otimes_{|\pi|} \ell_2^n,$$

and hence by the duality formula (9) we obtain

$$\begin{aligned}\big(\ell_2^n \otimes_{|\pi|} \ell_2^n\big)^* &= \big(\big(\ell_\infty^n \otimes_{|\pi|} \ell_\infty^n\big)^{\frac{1}{2}}\big(\ell_1^n \otimes_{|\pi|} \ell_1^n\big)^{\frac{1}{2}}\big)^*\\ &= \big(\ell_\infty^n \otimes_{|\pi|} \ell_\infty^n\big)^{*\frac{1}{2}}\big(\ell_1^n \otimes_{|\pi|} \ell_1^n\big)^{*\frac{1}{2}}\\ &= \big(\ell_\infty^n \otimes_{|\pi|} \ell_\infty^n\big)^{*\frac{1}{2}}\big(\ell_1^{n^2}\big)^{*\frac{1}{2}}\\ &= \big(\ell_\infty^n \otimes_{|\pi|} \ell_\infty^n\big)^{*\frac{1}{2}}\big(\ell_\infty^{n^2}\big)^{\frac{1}{2}} = \big(\ell_\infty^n \otimes_{|\pi|} \ell_\infty^n\big)^{*2},\end{aligned}$$

where the last equality is a consequence of (4). Now taking the 1/2-convexification we conclude by the isometric representation from (10) that with constants independent of n

$$\mathscr{L}^r\big(\ell_2^n,\ell_2^n\big)^{\frac{1}{2}}=\big(\ell_2^n\otimes_{|\pi|}\ell_2^n\big)^{*\frac{1}{2}}=\big(\ell_\infty^n\otimes_{|\pi|}\ell_\infty^n\big)^{*}=\mathscr{L}^r\big(\ell_\infty^n,\ell_1^n\big). \tag{12}$$

Considering diagonal operators we deduce that $\mathscr{L}^r(\ell_2^n,\ell_2^n)=(\ell_2^n\otimes_{|\pi|}\ell_2^n)^*$ contains an order isometric copy of ℓ_∞^n. Since $\ell_\infty^n=(\ell_\infty^n)^{\frac{1}{2}}$ holds isometrically, the space $\mathscr{L}^r(\ell_2^n,\ell_2^n)^{\frac{1}{2}}$ also contains ℓ_∞^n isometrically. On the other hand it follows from [17] that $\mathscr{L}^r(\ell_\infty,\ell_1)$ is an AL-space. This is a contradiction by the well-known fact that AL-spaces have Rademacher cotype 2. $\square$

5. Calderón–Lozanovskii interpolation of Fremlin tensor products

In the following section we show that our atomic approach can be extended to the more abstract setting of Calderón–Lozanovskii interpolation. For this we have to introduce some more notions. Fix a concave function $\varphi:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ which is homogeneous of degree one (i.e., $\varphi(\lambda s,\lambda t)=\lambda\varphi(s,t)$ for all $\lambda,s,t\geqslant 0$).

Following Lozanovskii [11], for any couple (X_0,X_1) of Banach lattices on a measure space (Ω,μ), we define the space $\varphi(X_0,X_1)$ of all $x\in L_0(\mu)$ such that $|x|=\varphi(|x_0|,|x_1|)$ for some $x_j\in X_j$, $j=0,1$. We note that $\varphi(X_0,X_1)$ is a Banach lattice equipped with the norm

$$\|x\|=\inf\big\{\max\big\{\|x_0\|_{X_0},\|x_1\|_{X_1}\big\}:\ |x|=\varphi\big(|x_0|,|x_1|\big),\ x_j\in X_j,\ j=0,1\big\}.$$

Properties of Banach lattices $\varphi(X_0,X_1)$ have been studied by Lozanovskii [11] (see also [13]).

Proposition 5.1. *Let (X_0,X_1) and (Y_0,Y_1) be couples of finite dimensional Banach lattices. Then*

(i) *If φ is super-multiplicative, i.e., there is $C_1>0$ such that*

$$\varphi(s,1)\varphi(t,1)\leqslant C_1\varphi(st,1),\quad s,t>0,$$

then we have

$$\big\|\mathrm{id}:\varphi(X_0,X_1)\otimes_{|\pi|}\varphi(Y_0,Y_1)\to\varphi(X_0\otimes_{|\pi|}Y_0,X_1\otimes_{|\pi|}Y_1)\big\|\leqslant C_1.$$

(ii) *If φ is sub-multiplicative, i.e., there is $C_2>0$ such that*

$$\varphi(st,1)\leqslant C_2\varphi(s,1)\varphi(t,1),\quad s,t>0,$$

then for $C=\max_{k=0,1}C(X_k\otimes_{|\pi|}Y_k)$ we have

$$\big\|\mathrm{id}:\varphi(X_0\otimes_{|\pi|}Y_0,X_1\otimes_{|\pi|}Y_1)\to\varphi(X_0,X_1)\otimes_{|\pi|}\varphi(Y_0,Y_1)\big\|\leqslant C_2C.$$

Since for each $0<\theta<1$ the function $\varphi(s,t)=s^{1-\theta}t^{\theta}$ with $s,t\geqslant 0$ is both super-multiplicative and sub-multiplicative with constant 1, this result extends Theorem 3.1 (for $m=2$) and Lemma 4.3.

Proof. In order to prove (i) note first that since φ is concave, for each choice of finitely many $s_1, \ldots, s_n \geqslant 0$ and $t_1, \ldots, t_n \geqslant 0$ we have

$$\sum_{j=1}^{n} \varphi(s_j, t_j) \leqslant \varphi\left(\sum_{j=1}^{n} s_j, \sum_{j=1}^{n} t_j\right).$$

Take $u \in \varphi(X_0, X_1) \otimes_{|\pi|} \varphi(Y_0, Y_1)$. Then for given $\varepsilon > 0$ we have

$$|u| \leqslant \sum_{j=1}^{n} x_j \otimes y_j$$

with positive $x_j \in \varphi(X_0, X_1)$ and $y_j \in \varphi(Y_0, Y_1)$ such that

$$\sum_{j=1}^{n} \|x_j\|_{\varphi(X_0,X_1)} \|y_j\|_{\varphi(Y_0,Y_1)} \leqslant (1+\varepsilon)\|u\|_{\varphi(X_0,X_1)\otimes_{|\pi|}\varphi(Y_0,Y_1)}.$$

By the definition of $\varphi(X_0, X_1)$ and $\varphi(Y_0, Y_1)$, it follows that for each $1 \leqslant j \leqslant n$ and $k = 0, 1$

$$\begin{aligned} x_j &\leqslant \varphi\big(x_j^0, x_j^1\big) \quad \text{with} \quad \big\|x_j^k\big\|_{X_k} \leqslant (1+\varepsilon)\|x_j\|_{\varphi(X_0,X_1)}, \\ y_j &\leqslant \varphi\big(y_j^0, y_j^1\big) \quad \text{with} \quad \big\|y_j^k\big\|_{Y_k} \leqslant (1+\varepsilon)\|y_j\|_{\varphi(Y_0,Y_1)}. \end{aligned}$$

Combining the above inequalities with the super-multiplicativity of φ yields that

$$|u| \leqslant \sum_{j=1}^{n} \varphi\big(x_j^0, x_j^1\big)\varphi\big(y_j^0, y_j^1\big) \leqslant C_1 \sum_{j=1}^{n} \varphi\big(x_j^0 \otimes y_j^0, x_j^1 \otimes y_j^1\big) \leqslant \varphi(u_0, u_1),$$

where $u_k := C_1 \sum_{j=1}^{n} x_j^k \otimes y_j^k$ for $k = 0, 1$. Note now that for $k = 0, 1$ we have

$$\begin{aligned} \|u_k\|_{X_k\otimes_{|\pi|}Y_k} &\leqslant C_1 \sum_{j=1}^{n} \big\|x_j^k \otimes y_j^k\big\|_{X_k\otimes_{|\pi|}Y_k} \leqslant C_1 \sum_{j=1}^{n} \big\|x_j^k\big\|_{X_k} \big\|y_j^k\big\|_{Y_k} \\ &\leqslant C_1(1+\varepsilon)^2 \sum_{j=1}^{n} \|x_j\|_{\varphi(X_0,X_1)} \|y_j\|_{\varphi(Y_0,Y_1)}. \end{aligned}$$

This all together implies that

$$\|u\|_{\varphi(X_0\otimes_{|\pi|}Y_0, X_1\otimes_{|\pi|}Y_1)} \leqslant C_1(1+\varepsilon)^3 \|u\|_{\varphi(X_0,X_1)\otimes_{|\pi|}\varphi(Y_0,Y_1)}.$$

Since ε was arbitrary, we obtain the required norm estimate from (i). The proof of (ii) is very similar to that of Lemma 4.3 and so we omit it. □

We conclude this section with the following remark: Combining (10) with Lozanovskii's remarkable duality formula (see [11,13])

$$\varphi(X_0, X_1)^* = \widehat{\varphi}\big(X_0^*, X_1^*\big),$$

where $\widehat{\varphi}(s,t) = \inf_{a,b>0} \frac{as+bt}{\varphi(a,b)}$, the preceding proposition can be reformulated in terms of spaces $\mathscr{L}^r(X_k, Y_k)$ of regular operators instead of Fremlin tensor products $X_k \otimes_{|\pi|} Y_k$. Note that the above Lozanovskii's formula holds true for any couple of finite dimensional lattices with universal constants.

6. Fremlin interpolation formulas – the multilinear case

So far we studied interpolation of Fremlin tensor products generated by pairs of finite dimensional Banach lattices:

$$[X_0 \otimes_{|\pi|} Y_0, X_1 \otimes_{|\pi|} Y_1]_\theta = [X_0, X_1]_\theta \otimes_{|\pi|} [Y_0, Y_1]_\theta, \quad 0 < \theta < 1.$$

In this section we are interested in such formulas for Fremlin tensor products which now are generated by m-tuples of finite dimensional Banach lattices:

$$\bigotimes\nolimits_{|\pi|}[X_k, Y_k]_\theta = \big[\bigotimes\nolimits_{|\pi|} X_k, \bigotimes\nolimits_{|\pi|} Y_k\big]_\theta, \quad 0 < \theta < 1.$$

As we have shown, in the case when $m = 2$ our results imply interpolation formulas for spaces of regular operators, and similarly such formulas for the case $m > 2$ lead to corresponding interpolation results for spaces of m-regular forms.

The main result of this section is the following.

Theorem 6.1. *Let (X_k, Y_k), $1 \leqslant k \leqslant m$ be couples of finite dimensional Banach lattices, and $1 < r_k < \infty$ such that $\sum_{k=1}^m \frac{1}{r_k} \leqslant 1$. Then*

(i) *If $M^{r_k}(X_k) = M^{r_k}(Y_k) = 1$ for all $1 \leqslant k \leqslant m$, then for every $0 < \theta < 1$ with equality of norms*

$$\big[\bigotimes\nolimits_{|\pi|} X_k, \bigotimes\nolimits_{|\pi|} Y_k\big]_\theta = \bigotimes\nolimits_{|\pi|}[X_k, Y_k]_\theta.$$

(ii) *If $M_{r_k}(X_k) = M_{r_k}(Y_k) = 1$ for all $1 \leqslant k \leqslant m$, then for every $0 < \theta < 1$ with equality of norms*

$$\big[\mathscr{B}^r(X_1, \ldots, X_m), \mathscr{B}^r(Y_1, \ldots, Y_m)\big]_\theta = \mathscr{B}^r\big([X_1, Y_1]_\theta, \ldots, [X_m, Y_m]_\theta\big).$$

The proof of this theorem is based on some results of independent interest we prove below. At first we recall the following classical inequality for positive operators (see, e.g., [10, p. 55]): If $T : X \to Y$ is a positive linear operator between Banach lattices, then for every $1 \leqslant p \leqslant \infty$ and every choice of $x_1, \ldots, x_n \in X$, we have

$$\Bigg\| \Bigg(\sum_{j=1}^n |Tx_j|^p \Bigg)^{1/p} \Bigg\|_Y \leqslant \|T\| \Bigg\| \Bigg(\sum_{j=1}^n |x_j|^p \Bigg)^{1/p} \Bigg\|_X. \tag{13}$$

The following multilinear extension of this result seems to be interesting on its own.

Theorem 6.2. *Let $T : X_1 \times \cdots \times X_n \to Y$ be a positive n-linear mapping in Banach lattices, and $1 \leqslant p \leqslant \infty$ such that $1/p = 1/p_1 + \cdots + 1/p_n$ with $1 \leqslant p_k \leqslant \infty$ for $k = 1, \ldots, n$. Then, for every choice of finitely many sequences $\{x_j^{(i)}\}_{j=1}^k$ in X_i, $1 \leqslant i \leqslant n$, we have*

$$\left\| \left(\sum_{j=1}^k \left| T\left(x_j^{(1)}, \ldots, x_j^{(n)}\right) \right|^p \right)^{1/p} \right\|_Y \leqslant \|T\| \left\| \left(\sum_{j=1}^k \left| x_j^{(1)} \right|^{p_1} \right)^{1/p_1} \right\|_{X_1} \cdots \left\| \left(\sum_{j=1}^k \left| x_j^{(n)} \right|^{p_n} \right)^{1/p_n} \right\|_{X_n}.$$

Before we give the proof of Theorem 6.2, we state below a result which will be used in the proof. Notice that the result is presented in [8, Corollary 3.6] in the case when $n = 2$. Similar arguments work for $n > 2$.

Lemma 6.3. *Let $\phi : C(K_1) \times \cdots \times C(K_n) \to \mathbb{R}$ be a positive n-linear form where $K_1, \ldots, K_n$ are compact Hausdorff spaces. Then*

(1) *There is a unique positive linear functional $h : C(K_1 \times \cdots \times K_n) \to \mathbb{R}$ such that for all $(f_1, \ldots, f_n) \in C(K_1) \times \cdots \times C(K_n)$,*

$$h(f_1 \otimes \cdots \otimes f_n) = \phi(f_1, \ldots, f_n).$$

(2) *There is a Radon measure μ on $K_1 \times \cdots \times K_n$ such that for all $(f_1, \ldots, f_n) \in C(K_1) \times \cdots \times C(K_n)$,*

$$\phi(f_1, \ldots, f_n) = \int (f_1 \otimes \cdots \otimes f_n)\, d\mu.$$

Proof of Theorem 6.2. Without loss of generality we may assume that all involved lattices are real, and $\|T\| = 1$. Fix finite sequences $\{x_j^{(i)}\}_{j=1}^k$ in X_i, $i = 1, \ldots, n$. Let y^* be a positive linear functional on Y with $\|y^*\|_{Y^*} \leqslant 1$. Since T is a positive operator, $|T(x_j^{(1)}, \ldots, x_j^{(n)})| \leqslant T(|x_j^{(1)}|, \ldots, |x_j^{(n)}|)$ for each $1 \leqslant j \leqslant k$. This implies

$$y^* \left(\sum_{j=1}^k \left| T\left(x_j^{(1)}, \ldots, x_j^{(n)}\right) \right|^p \right)^{1/p} \leqslant \left(\sum_{j=1}^k y^*\left(T\left(\left|x_j^{(1)}\right|, \ldots, \left|x_j^{(n)}\right| \right) \right)^p \right)^{1/p}.$$

Without loss of generality we may assume that $u_i \neq 0$ for each $1 \leqslant i \leqslant n$, where

$$u_i = \left(\sum_{j=1}^k \left| x_j^{(i)} \right|^{p_i} \right)^{1/p_i}.$$

For each $1 \leqslant i \leqslant n$, let $I(u_i)$ be the linear span of the order interval $[-u_i, u_i]$ in X_i. Taking $[-u_i, u_i]$ as the unit ball of $I(u_i)$, $I(u_i)$ is an abstract M-space, and by the well-known theorem of Kakutani (see, e.g., [10, p. 13]) there is a compact Hausdorff spaces K_i so that $I(u_i)$ is isometrically lattice isomorphic to $C(K_i)$. Let $J_i : C(K_i) \to X_i$ be the associated lattice isomorphism which maps the unit ball of $C(K_i)$ onto the order interval $[-u_i, u_i]$. Now define a positive and n-linear form ϕ on the Cartesian product $C(K_1) \times \cdots \times C(K_n)$ by

$$\phi(f_1, \dots, f_n) = y^*\big(T\big(J_1(f_1), \dots, J_n(f_n)\big)\big), \quad (f_1, \dots, f_n) \in C(K_1) \times \cdots \times C(K_n).$$

Then by Lemma 6.3, we conclude that there is a positive functional

$$h : C(K_1 \times \cdots \times K_n) \to \mathbb{R}$$

such that for every $f_1 \in C(K_1), \dots, f_n \in C(K_n)$ we have

$$\phi(f_1, \dots, f_n) = h(f_1 \otimes \cdots \otimes f_n).$$

Since $x_j^{(i)} \in [-u_i, u_i]$ for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$, there are positive functions $f_j^{(i)} \in C(K_i)$ with $\|f_j^{(i)}\|_{C(K_i)} \leqslant 1$ and $|x_j^{(i)}| = J_i(f_j^{(i)})$. Combining these remarks with the classical inequality for positive operators mentioned above, we obtain

$$\begin{aligned}
y^*\left(\sum_{j=1}^k \big|T\big(x_j^{(1)}, \dots, x_j^{(n)}\big)\big|^p\right)^{1/p} &\leqslant \left(\sum_{j=1}^k \phi\big(f_j^{(1)}, \dots, f_j^{(n)}\big)^p\right)^{1/p} \\
&\leqslant \left(\sum_{j=1}^k h\big(f_j^{(1)} \otimes \cdots \otimes f_j^{(n)}\big)^p\right)^{1/p} \\
&\leqslant \left\|\left(\sum_{j=1}^k f_j^{(1)} \otimes \cdots \otimes f_j^{(n)}\right)^p\right\|_{C(K_1 \times \cdots \times K_n)}.
\end{aligned}$$

Since all J_i are isometrical lattice isomorphisms, we have

$$\begin{aligned}
\left\|\left(\sum_{j=1}^k \big|f_j^{(i)}\big|^{p_i}\right)^{1/p_i}\right\|_{C(K_i)} &= \left\|\left(\sum_{j=1}^k J_i^{-1}\big(\big|x_j^{(i)}\big|\big)^{p_i}\right)^{1/p_i}\right\|_{C(K_i)} \\
&\leqslant \left\|\left(\sum_{j=1}^k \big|x_j^{(i)}\big|^{p_i}\right)^{1/p_i}\right\|_{X_i}.
\end{aligned}$$

Consequently, by Hölder's inequality we get

$$\begin{aligned}
&y^*\left(\sum_{j=1}^k \big|T\big(x_j^{(1)}, \dots, x_j^{(n)}\big)\big|^p\right)^{1/p} \\
&\quad\leqslant \left\|\left(\sum_{j=1}^k \big|f_j^{(1)}\big|^{p_1}\right)^{1/p_1} \otimes \cdots \otimes \left(\sum_{j=1}^k \big|f_j^{(n)}\big|^{p_n}\right)^{1/p_n}\right\|_{C(K_1 \times \cdots \times K_n)}
\end{aligned}$$

$$= \left\| \left(\sum_{j=1}^{k} \left| f_j^{(1)} \right|^{p_1} \right)^{1/p_1} \right\|_{C(K_1)} \cdots \left\| \left(\sum_{j=1}^{k} \left| f_j^{(n)} \right|^{p_n} \right)^{1/p_n} \right\|_{C(K_n)}$$
$$\leqslant \left\| \left(\sum_{j=1}^{k} \left| x_j^{(1)} \right|^{p_1} \right)^{1/p_1} \right\|_{X_1} \cdots \left\| \left(\sum_{j=1}^{k} \left| x_j^{(n)} \right|^{p_n} \right)^{1/p_n} \right\|_{X_n}.$$

Since $\|y\|_Y = \sup_{\|z^*\|_{Y^*} \leqslant 1} |z^*(y)|$ for every $y \in Y$, the proof completes. $\square$

Using Theorem 6.2 and the lattice variants of the Maurey–Rosenthal factorization theorem from [3] we are able to prove the following.

Lemma 6.4. *Let (X_k, Y_k), $1 \leqslant k \leqslant m$, be couples of finite dimensional Banach lattices, and $1 < r_k < \infty$ such that $\sum_{k=1}^{m} \frac{1}{r_k} \leqslant 1$. Then for each $0 < \theta < 1$ we have*

$$\left\| \mathrm{id} : \left[\bigotimes_{|\pi|} X_k, \bigotimes_{|\pi|} Y_k \right]_\theta \to \bigotimes_{|\pi|} [X_k, Y_k]_\theta \right\| \leqslant \prod_{k=1}^{m} M^{r_k}(X_k)^{1-\theta} M^{r_k}(Y_k)^{\theta}$$

and

$$\left\| \mathrm{id} : \mathscr{B}^r\big([X_1, Y_1]_\theta, \ldots, [X_m, Y_m]_\theta\big) \to \left[\mathscr{B}^r(X_1, \ldots, X_m), \mathscr{B}^r(Y_1, \ldots, Y_m) \right]_\theta \right\|$$
$$\leqslant \prod_{k=1}^{m} M^{r_k}(X_k)^{1-\theta} M^{r_k}(Y_k)^{\theta}.$$

Proof. Without loss of generality (see again (6)) we may assume that $M^{r_k}(X_k) = M^{r_k}(Y_k) = 1$. The duality formula (11) shows that it is enough to prove the second norm estimate on regular forms. Fix $\Phi \in \mathscr{B}^r([X_1, Y_1]_\theta, \ldots, [X_m, Y_m]_\theta)$, and assume without loss of generality that Φ is positive. From Theorem 6.2, it follows that Φ for all choices of vectors $x_1^{(j)}, \ldots, x_k^{(j)} \in [X_j, Y_j]_\theta$, $1 \leqslant j \leqslant m$, satisfies the following inequality:

$$\left(\sum_{j=1}^{k} \left| \Phi\big(x_j^{(1)}, \ldots, x_j^{(m)}\big) \right|^r \right)^{1/r}$$
$$\leqslant \|\Phi\| \left\| \left(\sum_{j=1}^{k} \left| x_j^{(1)} \right|^{r_1} \right)^{1/r_1} \right\|_{[X_1, Y_1]_\theta} \cdots \left\| \left(\sum_{j=1}^{k} \left| x_j^{(m)} \right|^{r_m} \right)^{1/r_m} \right\|_{[X_m, Y_m]_\theta},$$

where $\frac{1}{r} = \sum_{k=1}^{m} \frac{1}{r_k} \leqslant 1$. It follows from (7) that

$$M^{r_k}\big([X_k, Y_k]_\theta\big) \leqslant M^{r_k}(X_k)^{1-\theta} M^{r_k}(Y_k)^{\theta} = 1,$$

and so we conclude from [3, Theorem 1] that there exist positive diagonal operators $D_{\lambda_k} : [X_k, Y_k]_\theta \to \ell_{r_k}^{n_k}$ with $n_k = \dim X_k = \dim Y_k$ and a positive form $R : \prod_{k=1}^{m} \ell_{r_k}^{n_k} \to \mathbb{C}$ such that the following diagram commutes

$$
\begin{tikzcd}
\prod_{k=1}^m [X_k, Y_k]_\theta \arrow[r, "\Phi"] \arrow[d, "\times_k D_{\lambda_k}"'] & \mathbb{C} \\
\prod_{k=1}^m \ell_{r_k}^{n_k} \arrow[ur, "R"'] &
\end{tikzcd}
$$

and moreover $\|R\| \leqslant \|\Phi\|$, $\|D_{\lambda_k}\| \leqslant 1$ for $1 \leqslant k \leqslant m$. Define the map

$$\Psi : \prod_{k=1}^m \mathscr{M}\big(X_k, \ell_{r_k}^{n_k}\big) + \prod_{k=1}^m \mathscr{M}\big(Y_k, \ell_{r_k}^{n_k}\big) \to \mathscr{B}^r(X_1, \ldots, X_m) + \mathscr{B}^r(Y_1, \ldots, Y_m)$$

by

$$\Psi(D_{g_1}, \ldots, D_{g_m}) = R \circ (\times_k D_{g_k})$$

for $(D_{g_1}, \ldots, D_{g_m}) \in \prod_{k=1}^m \mathscr{M}(X_k, \ell_{r_k}^{n_k}) + \prod_{k=1}^m \mathscr{M}(Y_k, \ell_{r_k}^{n_k})$ (here $\mathscr{M}(X_k, \ell_{r_k}^{n_k})$ stands for the Banach space of all diagonal operators). Clearly

$$\Psi : \left(\prod_{k=1}^m \mathscr{M}\big(X_k, \ell_{r_k}^{n_k}\big), \prod_{k=1}^m \mathscr{M}\big(Y_k, \ell_{r_k}^{n_k}\big)\right) \to \big(\mathscr{B}^r(X_1, \ldots, X_m), \mathscr{B}^r(Y_1, \ldots, Y_m)\big)$$

and

$$\left\|\Psi : \prod_{k=1}^m \mathscr{M}\big(X_k, \ell_{r_k}^{n_k}\big) \to \mathscr{B}^r(X_1, \ldots, X_m)\right\| \leqslant \|R\|,$$
$$\left\|\Psi : \prod_{k=1}^m \mathscr{M}\big(Y_k, \ell_{r_k}^{n_k}\big) \to \mathscr{B}^r(Y_1, \ldots, Y_m)\right\| \leqslant \|R\|.$$

Then by the complex multilinear interpolation theorem we obtain

$$\left\|\Psi : \prod_{k=1}^m \big[\mathscr{M}\big(X_k, \ell_{r_k}^{n_k}\big), \mathscr{M}\big(Y_k, \ell_{r_k}^{n_k}\big)\big]_\theta \to \big[\mathscr{B}^r(X_1, \ldots, X_m), \mathscr{B}^r(Y_1, \ldots, Y_m)\big]_\theta\right\| \leqslant \|R\|.$$

As in [7, Lemma 4], it follows from [5, Proposition 3.5] that with equality of norms we have

$$\begin{aligned}
\big[\mathscr{M}\big(X_k, \ell_{r_k}^{n_k}\big), \mathscr{M}\big(Y_k, \ell_{r_k}^{n_k}\big)\big]_\theta &= \big[\big(\big(X_k^{r_k}\big)^\times\big)^{1/r_k}, \big(\big(Y_k^{r_k}\big)^\times\big)^{1/r_k}\big]_\theta \\
&= \big(\big(\big(X_k^{r_k}\big)^\times\big)^{1/r_k}\big)^{1-\theta}\big(\big(\big(Y_k^{r_k}\big)^\times\big)^{1/r_k}\big)^\theta = \big(\big(\big(X_k^{1-\theta}Y_k^\theta\big)^{r_k}\big)^\times\big)^{1/r_k} \\
&= \mathscr{M}\big([X_k, Y_k]_\theta, \ell_{r_k}^{n_k}\big),
\end{aligned}$$

where as usual, $X^\times$ denotes the Köthe dual of a Banach lattice X. Since $\|R\| \leqslant \|\Phi\|$, we finally obtain

$$\|\Phi\|_{[\mathscr{B}^r(X_1,\dots,X_m),\mathscr{B}^r(Y_1,\dots,Y_m)]_\theta} = \big\|\Psi(D_{\lambda_1},\dots,D_{\lambda_m})\big\|_{[\mathscr{B}^r(X_1,\dots,X_m),\mathscr{B}^r(Y_1,\dots,Y_m)]_\theta}$$
$$\leqslant \|R\|\prod_{k=1}^m \|D_{\lambda_k}\| \leqslant \|\Phi\|_{\mathscr{B}^r([X_1,Y_1]_\theta,\dots,[X_m,Y_m]_\theta)}. \qquad \square$$

Now notice that combining the above lemma with Theorem 3.1 easily gives the proof of the main result of this section already stated in Theorem 6.1. The following consequence of Theorem 6.1 and (5) is a counterpart of Corollary 4.4.

Corollary 6.5. *For $k = 1,\dots,m$ let $1 < p_k, q_k < \infty$, and let $1/s_k = (1-\theta)/p_k + \theta/q_k$ with $0 < \theta < 1$. Then*

(i) *If $\sum_{k=1}^m \frac{1}{\min\{p_k,q_k\}} \leqslant 1$, then for each n the following formula holds with equality of norms*

$$\big[\bigotimes_{|\pi|}\ell^n_{p_k}, \bigotimes_{|\pi|}\ell^n_{q_k}\big]_\theta = \bigotimes_{|\pi|}\ell^n_{s_k}.$$

(ii) *If $\sum_{k=1}^m \frac{1}{\min\{p_k,q_k\}} \leqslant 1$, then for each n the following formula holds with equality of norms*

$$\big[\mathscr{B}^r\big(\ell^n_{p_1},\dots,\ell^n_{p_m}\big), \mathscr{B}^r\big(\ell^n_{q_1},\dots,\ell^n_{q_m}\big)\big]_\theta = \mathscr{B}^r\big(\ell^n_{s_1},\dots,\ell^n_{s_m}\big).$$

7. Schur factorization of regular operators

In the final section we show applications of our interpolation formulas for Fremlin tensor products to Schur type factorization of matrices. At first, notice that a combination of the fundamental Riesz order isometry from (10) and Lemma 4.3 and Lemma 6.4 yields as an immediate consequence Theorem 1.2 which was already formulated in the introduction.

The following corollary is a consequence of (5) and Theorem 1.2.

Corollary 7.1. *Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and $0 < \theta < 1$, and define q_θ, p_θ as usual through $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$. Then for every matrix $A \in M_n$ with operator norm*

$$\big\||A| : \ell^n_{p_\theta} \to \ell^n_{q_\theta}\big\| \leqslant 1$$

and for every choice of $1 \leqslant r, s \leqslant \infty$ there exist matrices $A_0, A_1 \in M_n$ such that

$$|A| = |A_0|^{1-\theta} * |A_1|^\theta \tag{14}$$

and

$$\max_{k=0,1}\big\||A_k| : \ell^n_{p_k} \to \ell^n_{q_k}\big\| \leqslant n^{(\max\{0,\frac{1}{p_0}-\frac{1}{r}\}+\max\{0,\frac{1}{s}-\frac{1}{q_0}\})(1-\theta)+(\max\{0,\frac{1}{p_1}-\frac{1}{r}\}+\max\{0,\frac{1}{s}-\frac{1}{q_1}\})\theta}.$$

Alternatively, for $1 < r, s < \infty$ with $1/r + 1/s \leqslant 1$ we have

$$\max_{k=0,1}\big\||A_k| : \ell^n_{p_k} \to \ell^n_{q_k}\big\| \leqslant n^{(\max\{0,\frac{1}{p_0}-\frac{1}{r}\}+\max\{0,\frac{1}{s'}-\frac{1}{q_0}\})(1-\theta)+(\max\{0,\frac{1}{p_1}-\frac{1}{r}\}+\max\{0,\frac{1}{s'}-\frac{1}{q_1}\})\theta}.$$

Finally, we show how our Schur type factorization results for the regular norm can be modified to factorize matrices A as in (14), however the difference now is that we consider norm estimates on A_0 and A_1 with respect to certain variants of the regular norm.

Based on deep results of Kouba [9] it was proved in [7, Lemma 9] that for two finite dimensional interpolation couples (E_0, E_1) and (F_0, F_1) of Banach spaces we have

$$\begin{aligned} &\big\| \mathrm{id} : \Gamma_2\big([E_0, E_1]_\theta, [F_0, F_1]_\theta\big) \to \big[\Gamma_2(E_0, F_0), \Gamma_2(E_1, F_1)\big]_\theta \big\| \\ &\quad \leqslant \big(T_2(E_0)^{1-\theta} T_2(E_1)^\theta\big)\big(T_2\big(F_0^*\big)^{1-\theta} T_2\big(F_1^*\big)^\theta\big). \end{aligned} \tag{15}$$

Here $T_2(E)$ is the Rademacher type 2 constant of a Banach space E, and $\Gamma_2(E, F)$ denotes the space $\mathscr{L}(E, F)$ equipped with the hilbertian norm $\gamma_2(T) = \inf \|R\|\|S\|$, where the infimum is taken over all n and all possible factorizations of T through finite dimensional Hilbert spaces ℓ_2^n,

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "R"'] & F \\ \ell_2^n \arrow[ur, "S"'] & \end{tikzcd}$$

Similarly, we define for two finite dimensional Banach lattices X and Y the Banach lattice $\Gamma_2^r(E, F)$ to be $\mathscr{L}(E, F)$ equipped with the regularly hilbertian norm $\gamma_2^r(T) = \inf \|R\|_r \|S\|_r$, where the infimum is taken over all n and all operators $R \in \mathscr{L}(X, \ell_2^n)$ and $S \in \mathscr{L}(\ell_2^n, Y)$ such that $|T| \leqslant S \circ R$.

Theorem 7.2. *Let (X_0, X_1) and (Y_0, Y_1) be two finite dimensional interpolation couples of Banach lattices. Then for every $0 < \theta < 1$ we have*

$$\begin{aligned} &\big\| \mathrm{id} : \Gamma_2^r\big([X_0, X_1]_\theta, [Y_0, Y_1]_\theta\big) \to \big[\Gamma_2^r(X_0, Y_0), \Gamma_2^r(Y_0, Y_1)\big]_\theta \big\| \\ &\quad \leqslant \big(M^2(X_0)^{1-\theta} M^2(X_1)^\theta\big)\big(M_2(Y_0)^{1-\theta} M_2(Y_1)^\theta\big). \end{aligned}$$

Proof. Fix a positive integer n, and consider for $k = 0, 1$ the following contractions

$$\mathscr{L}^r\big(X_k, \ell_2^n\big) \otimes_{|\pi|} \mathscr{L}^r\big(\ell_2^n, Y_k\big) \to \Gamma_2^r(X_k, Y_k), \quad R \otimes S \mapsto S \circ R.$$

Then by complex interpolation we obtain

$$\begin{aligned} &\big\| \big[\mathscr{L}^r\big(X_0, \ell_2^n\big) \otimes_{|\pi|} \mathscr{L}^r\big(\ell_2^n, Y_0\big), \mathscr{L}^r\big(X_1, \ell_2^n\big) \otimes_{|\pi|} \mathscr{L}^r\big(\ell_2^n, Y_1\big)\big]_\theta \\ &\quad \to \big[\Gamma_2^r(X_0, Y_0), \Gamma_2^r(X_1, Y_1)\big]_\theta \big\| \leqslant 1. \end{aligned}$$

Now observe that for $k = 0, 1$ the canonical bilinear mappings

$$\mathscr{L}^r\big(X_k, \ell_2^n\big) \times \mathscr{L}^r\big(\ell_2^n, Y_k\big) \to \mathscr{L}^r\big(X_k, \ell_2^n\big) \otimes_{|\pi|} \mathscr{L}^r\big(\ell_2^n, Y_k\big)$$

are positive contractions. Then bilinear complex interpolation and extension to the Fremlin tensor product yields

$$\begin{aligned}&\big\|\big[\mathscr{L}^r\big(X_0,\ell_2^n\big),\mathscr{L}^r\big(X_1,\ell_2^n\big)\big]_\theta\otimes_{|\pi|}\big[\mathscr{L}^r\big(\ell_2^n,Y_0\big),\mathscr{L}^r\big(\ell_2^n,Y_1\big)\big]_\theta\\&\quad\to\big[\mathscr{L}^r\big(X_0,\ell_2^n\big)\otimes_{|\pi|}\mathscr{L}^r\big(\ell_2^n,Y_0\big),\mathscr{L}^r\big(X_1,\ell_2^n\big)\otimes_{|\pi|}\mathscr{L}^r\big(\ell_2^n,Y_1\big)\big]_\theta\big\|\leqslant 1.\end{aligned}$$

From Lemma 4.3, we obtain by duality

$$\begin{aligned}&\big\|\mathscr{L}^r\big([X_0,X_1]_\theta,\ell_2^n\big)\hookrightarrow\big[\mathscr{L}^r\big(X_0,\ell_2^n\big),\mathscr{L}^r\big(X_1,\ell_2^n\big)\big]_\theta\big\|\leqslant M^2(X_0)^{1-\theta}M^2(X_1)^\theta,\\&\big\|\mathscr{L}^r\big(\ell_2^n,[Y_0,Y_1]_\theta\big)\hookrightarrow\big[\mathscr{L}^r\big(\ell_2^n,Y_0\big),\mathscr{L}^r\big(\ell_2^n,Y_1\big)\big]_\theta\big\|\leqslant M_2(Y_0)^{1-\theta}M_2(Y_1)^\theta.\end{aligned}$$

In consequence the above estimates give

$$\begin{aligned}&\big\|\mathscr{L}^r\big([X_0,X_1]_\theta,\ell_2^n\big)\otimes_{|\pi|}\mathscr{L}^r\big(\ell_2^n,[Y_0,Y_1]_\theta\big)\to\big[\Gamma_2^r(X_0,Y_0),\Gamma_2^r(X_1,Y_1)\big]_\theta\big\|\\&\quad\leqslant\big(M^2(X_0)^{1-\theta}M^2(X_1)^\theta\big)\big(M_2(Y_0)^{1-\theta}M_2(Y_1)^\theta\big).\end{aligned}$$

To finish the proof let us fix $T\in\Gamma_2^r([X_0,X_1]_\theta,[Y_0,Y_1]_\theta)$. Then for a given $\varepsilon>0$ there exist n and

$$R\otimes S\in\mathscr{L}^r\big([X_0,X_1]_\theta,\ell_2^n\big)\otimes_{|\pi|}\mathscr{L}^r\big(\ell_2^n,[Y_0,Y_1]_\theta\big)$$

which satisfy $|T|\leqslant S\circ R$ and

$$\|R\otimes S\|_{\mathscr{L}^r([X_0,X_1]_\theta,\ell_2^n)\otimes_{|\pi|}\mathscr{L}^r(\ell_2^n,[Y_0,Y_1]_\theta)}\leqslant(1+\varepsilon)\|T\|_{\Gamma_2^r([X_0,X_1]_\theta,[Y_0,Y_1]_\theta)}.$$

Finally, we get with $M:=[M^2(X_0)^{1-\theta}M^2(X_1)^\theta][M_2(Y_0)^{1-\theta}M_2(Y_1)^\theta]$ that

$$\begin{aligned}\|T\|_{[\Gamma_2^r(X_0,Y_0),\Gamma_2^r(X_1,Y_1)]_\theta}&\leqslant M\|R\otimes S\|_{\mathscr{L}^r([X_0,X_1]_\theta,\ell_2^n)\otimes_{|\pi|}\mathscr{L}^r(\ell_2^n,[Y_0,Y_1]_\theta)}\\&\leqslant(1+\varepsilon)M\|T\|_{\Gamma_2^r([X_0,X_1]_\theta,[Y_0,Y_1]_\theta)}.\end{aligned}$$

Since ε was arbitrary, this completes the proof. □

As a consequence (use also (5)), we obtain the following corollary in finite dimensional ℓ_p-spaces (compare with Corollaries 4.4 and 6.5).

Corollary 7.3. *Let $1\leqslant s_0,s_1\leqslant 2\leqslant r_0,r_1\leqslant\infty$ and $0<\theta<1$, and define s_θ,r_θ as in Corollary* 7.1. *Moreover, let $A\in M_n$ be a matrix such that there are $R,S\in M_n$ with*

$$|A|\leqslant S\circ R\quad\textit{and}\quad\big\||R|:\ell_{r_\theta}^n\to\ell_2^n\big\|,\big\||S|:\ell_2^n\to\ell_{s_\theta}^n\big\|\leqslant 1.$$

Then there are matrices $A_0,A_1,R_0,R_1,S_0,S_1\in M_n$ such that

$$|A|=|A_0|^{1-\theta}*|A_1|^\theta$$

and for $j=0,1$ we have $|A_j|\leqslant S_j\circ R_j$ with

$$\big\||R_j|:\ell_{r_j}^n\to\ell_2\big\|\leqslant 1,\qquad\big\||S_j|:\ell_2\to\ell_{s_j}^n\big\|\leqslant 1.$$

We conclude with the remark that similarly to the class of regularly hilbertian operators we can define regularly (p, q)-factorable operators in Banach lattices (for such operators in Banach spaces see, e.g., [4]). Then the techniques we used prove interpolation formulas for these classes, or equivalently Schur factorization theorems for matrices under restrictions of their regularly (p, q)-factorable norms. Finally, we mention that via duality Corollary 7.3 leads to an interpolation result on a certain class of summing operators (which according to the language of the theory of Banach operator ideals should be called regularly 2-dominated operators).

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